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A Study on Fractional L'Hospital's Rule

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***Abstract*: This paper studies the fractional L'Hospital's rule based on Jumarie type of Riemann-Liouville (R-L) fractional derivative. A new multiplication of fractional analytic functions plays an important role in this article. On the other hand, we provide some limit problems to illustrate the applications of fractional L'Hospital's rule. In fact, these results we obtained are generalizations of those in ordinary calculus.**

***Keyword*: Fractional L'Hospital's rule, Jumarie type of R-L fractional derivative, New multiplication, Fractional analytic functions, Limit problems.**

I. INTRODUCTION

During the 18th and 19th centuries, there were many famous scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and some others who reported interesting results within fractional calculus. In recent years, fractional calculus has become an increasingly popular research area due to its effective applications in different scientific fields such as economics, engineering mathematics, dynamics, mathematical biology, control theory, optimization, chaos theory, quantum mechanics, and so on [1-9]. Fractional calculus continues to rapidly develop with different definitions of derivatives and integrals [10-13]. New fractional derivatives and integrals have become some of the most effective tools for contributing to physical phenomena. They can also be used as applications for real-life problems.

In this paper, based on Jumarie type of R-L fractional derivative, the fractional L'Hospital's rule is studied. We can prove the fractional L'Hospital's rule by using two major methods: a new multiplication of fractional analytic functions and fractional Taylor series. In addition, we also provide some limit problems to illustrate the applications of fractional L'Hospital's rule. In fact, our results are generalizations of those in classical calculus.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper and some important properties.

Definition 2.1 ([14])**:** Suppose that $0 < \alpha \leq 1$, and x_0 is a real number. The Jumarie's modified R-L α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt. \tag{1}$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([15])**:** *Let* α, β, x_0, C *be real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Definition 2.3 ([16])**:** Assume that x, x_0 and p_k are real numbers for all k, and let $0 < \alpha \leq 1$. Then the series $\sum_{k=0}^{\infty} p_k (x - x_0)^{k\alpha}$ is called a real α-fractional power series. Its disk of convergence intersects the real axis in an interval $(x_0 - r, x_0 + r)$ called the interval of convergence. Each α-fractional power series defines a real valued sum function whose value at each x in the interval of convergence is given by

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} p_k (x - x_0)^{k\alpha} . \tag{4}$$

This series is said to represent f_α in the interval of convergence, and it is called a α-fractional power series expansion of f_α about x_0.

Definition 2.4 ([16])**:** If $0 < \alpha \leq 1$ and let f_α be a real valued α-fractional function defined on an interval I contained in R. If f_α has α-fractional derivatives of every order at each point of I, we write $f_\alpha \in C_\alpha^\infty(I)$. If $f_\alpha \in C_\alpha^\infty(I)$ on some neighborhood of a point x_0, the series

$$\sum_{k=0}^{\infty} \frac{\left({}_{x_0}D_x^\alpha\right)^k [f(x)](x_0)}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha} \tag{5}$$

is called the α-fractional Taylor series about x_0 generated by f_α. To indicate that f_α generate this fractional Taylor series, we write

$$f_\alpha(x^\alpha) \sim \sum_{k=0}^{\infty} \frac{\left({}_{x_0}D_x^\alpha\right)^k [f(x)](x_0)}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha}. \tag{6}$$

Theorem 2.5 ([16])**:** *Let* $0 < \alpha \leq 1$, *and* $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} c_k (x - x_0)^{k\alpha}$. *Then*

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{\left({}_{x_0}D_x^\alpha\right)^k [f(x)](x_0)}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha}. \tag{7}$$

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.6 ([17])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes k}, \tag{8}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes k}. \tag{9}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha} \\ & = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right) (x - x_0)^{k\alpha}. \end{aligned} \tag{10}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes k} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right) \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes k}. \end{aligned} \tag{11}$$

Definition 2.7 ([17])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes k}, \tag{12}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes k}. \tag{13}$$

We define the compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \big(g_\alpha(x^\alpha)\big)^{\otimes k}, \tag{14}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \big(f_\alpha(x^\alpha)\big)^{\otimes k}. \tag{15}$$

Definition 2.8 ([17])**:** Let $0 < \alpha \le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions at x_0 satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha. \tag{16}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

The followings are some fractional analytic functions.

Definition 2.9 ([18])**:** Suppose that $0 < \alpha \le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{17}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

Definition 2.10 ([18])**:** The α-fractional cosine and sine function are defined respectively as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k\alpha+1)} x^{2k\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2k}, \tag{18}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma((2k+1)\alpha+1)} x^{(2k+1)\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (2k+1)}. \tag{19}$$

Definition 2.11 ([19])**:** Let $0 < \alpha \le 1$. If $u_\alpha(x^\alpha)$, $w_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then the α-fractional power exponential function $u_\alpha(x^\alpha)^{\otimes w_\alpha(x^\alpha)}$ is defined by

$$u_\alpha(x^\alpha)^{\otimes w_\alpha(x^\alpha)} = E_\alpha\left(w_\alpha(x^\alpha) \otimes Ln_\alpha\big(u_\alpha(x^\alpha)\big)\right). \tag{20}$$

III. MAIN RESULT AND APPLICATIONS

In this section, we will prove the fractional L'Hospital's rule and provide some examples to illustrate its applications.

Theorem 3.1 (fractional L'Hospital's rule)**:** *Assume that* $0 < \alpha \le 1$, c *is a real number, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ $[g_\alpha(x^\alpha)]^{\otimes -1}$ *are* α*-fractional analytic functions at* $x = c$. *If* $\lim_{x\to c} f_\alpha(x^\alpha) = \lim_{x\to c} g_\alpha(x^\alpha) = 0$, *or* $\lim_{x\to c} f_\alpha(x^\alpha) = \pm\infty$, *and* $\lim_{x\to c} g_\alpha(x^\alpha) = \pm\infty$. *Suppose that* $\lim_{x\to c} f_\alpha(x^\alpha) \otimes [g_\alpha(x^\alpha)]^{\otimes -1}$ *and* $\lim_{x\to c} \left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}$ *exist,* $\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c) \neq 0$. *Then*

$$\lim_{x\to c} f_\alpha(x^\alpha) \otimes [g_\alpha(x^\alpha)]^{\otimes -1} = \lim_{x\to c} \left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}. \tag{21}$$

Proof *Case 1*. If $\lim_{x\to c} f_\alpha(x^\alpha) = \lim_{x\to c} g_\alpha(x^\alpha) = 0$. By Theorem 2.5, we have

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{\left({}_cD_x^\alpha\right)^k [f_\alpha(x^\alpha)](c)}{\Gamma(k\alpha+1)} (x - c)^{k\alpha}$$

$$= \frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c)}{\Gamma(\alpha+1)} (x - c)^\alpha + \frac{\left({}_cD_x^\alpha\right)^2 [f_\alpha(x^\alpha)](c)}{\Gamma(2\alpha+1)} (x - c)^{2\alpha} + \cdots$$

$$= \frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c)}{1!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 1} + \frac{\left({}_cD_x^\alpha\right)^2[f_\alpha(x^\alpha)](c)}{2!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 2} + \cdots. \tag{22}$$

And

$$g_\alpha(x^\alpha) = \frac{\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c)}{\Gamma(\alpha+1)}(x-c)^\alpha + \frac{\left({}_cD_x^\alpha\right)^2[g_\alpha(x^\alpha)](c)}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + \cdots$$

$$= \frac{\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c)}{1!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 1} + \frac{\left({}_cD_x^\alpha\right)^2[g_\alpha(x^\alpha)](c)}{2!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 2} + \cdots. \tag{23}$$

Therefore,

$$\lim_{x\to c} f_\alpha(x^\alpha)\otimes[g_\alpha(x^\alpha)]^{\otimes -1}$$

$$= \lim_{x\to c} \left(\frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c)}{1!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 1} + \frac{\left({}_cD_x^\alpha\right)^2[f_\alpha(x^\alpha)](c)}{2!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 2} + \cdots\right)\otimes \left[\frac{\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c)}{1!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 1} + \frac{\left({}_cD_x^\alpha\right)^2[g_\alpha(x^\alpha)](c)}{2!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 2} + \cdots\right]^{\otimes -1}$$

$$= \lim_{x\to c} \left(\frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c)}{1!} + \frac{\left({}_cD_x^\alpha\right)^2[f_\alpha(x^\alpha)](c)}{2!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 1} + \cdots\right)\otimes \left[\frac{\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c)}{1!} + \frac{\left({}_cD_x^\alpha\right)^2[g_\alpha(x^\alpha)](c)}{2!}\left(\frac{1}{\Gamma(\alpha+1)}(x-c)^\alpha\right)^{\otimes 1} + \cdots\right]^{\otimes -1}$$

$$= \frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c)}{\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c)}. \tag{24}$$

On the other hand,

$$\lim_{x\to c}\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)]\otimes\left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}$$

$$= \lim_{x\to c} \left({}_cD_x^\alpha\right)\left[\left(\frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c)}{\Gamma(\alpha+1)}(x-c)^\alpha + \frac{\left({}_cD_x^\alpha\right)^2[f_\alpha(x^\alpha)](c)}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + \cdots\right)\right]\otimes \left[\left({}_cD_x^\alpha\right)\left[\frac{\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c)}{\Gamma(\alpha+1)}(x-c)^\alpha + \frac{\left({}_cD_x^\alpha\right)^2[g_\alpha(x^\alpha)](c)}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + \cdots\right]\right]^{\otimes -1}$$

$$= \lim_{x\to c} \left(\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c) + \frac{\left({}_cD_x^\alpha\right)^2[f_\alpha(x^\alpha)](c)}{\Gamma(\alpha+1)}(x-c)^\alpha + \cdots\right)\otimes \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c) + \frac{\left({}_cD_x^\alpha\right)^2[g_\alpha(x^\alpha)](c)}{\Gamma(2\alpha+1)}(x-c)^\alpha + \cdots\right]^{\otimes -1}$$

$$= \frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c)}{\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c)}. \tag{25}$$

Thus, the desired result holds.

Case 2. If $\lim_{x\to c} f_\alpha(x^\alpha) = \lim_{x\to c} g_\alpha(x^\alpha) = \pm\infty$. Then $\lim_{x\to c}[f_\alpha(x^\alpha)]^{\otimes -1} = \lim_{x\to c}[g_\alpha(x^\alpha)]^{\otimes -1} = 0$, and

$$\lim_{x\to c} f_\alpha(x^\alpha)\otimes[g_\alpha(x^\alpha)]^{\otimes -1}$$

$$= \lim_{x\to c}[g_\alpha(x^\alpha)]^{\otimes -1}\otimes[[f_\alpha(x^\alpha)]^{\otimes -1}]^{\otimes -1}$$

$$= \lim_{x \to c} \left({}_{c}D_{x}^{\alpha} \right) [[g_{\alpha}(x^{\alpha})]^{\otimes -1}] \otimes \left[\left({}_{c}D_{x}^{\alpha} \right) [[f_{\alpha}(x^{\alpha})]^{\otimes -1}] \right]^{\otimes -1} \text{ (by Case 1)}$$

$$= \lim_{x \to c} [g_{\alpha}(x^{\alpha})]^{\otimes -2} \otimes \left({}_{c}D_{x}^{\alpha} \right) [g_{\alpha}(x^{\alpha})] \otimes \left[[f_{\alpha}(x^{\alpha})]^{\otimes -2} \otimes \left({}_{c}D_{x}^{\alpha} \right) [f_{\alpha}(x^{\alpha})] \right]^{\otimes -1}$$

$$= \lim_{x \to c} [f_{\alpha}(x^{\alpha}) \otimes [g_{\alpha}(x^{\alpha})]^{\otimes -1}]^{\otimes 2} \otimes \left({}_{c}D_{x}^{\alpha} \right) [g_{\alpha}(x^{\alpha})] \otimes \left[\left({}_{c}D_{x}^{\alpha} \right) [f_{\alpha}(x^{\alpha})] \right]^{\otimes -1}$$

$$= \lim_{x \to c} [f_{\alpha}(x^{\alpha}) \otimes [g_{\alpha}(x^{\alpha})]^{\otimes -1}]^{\otimes 2} \cdot \lim_{x \to c} \left({}_{c}D_{x}^{\alpha} \right) [g_{\alpha}(x^{\alpha})] \otimes \left[\left({}_{c}D_{x}^{\alpha} \right) [f_{\alpha}(x^{\alpha})] \right]^{\otimes -1}. \tag{26}$$

Thus,

$$\lim_{x \to c} f_{\alpha}(x^{\alpha}) \otimes [g_{\alpha}(x^{\alpha})]^{\otimes -1} = \lim_{x \to c} \left({}_{c}D_{x}^{\alpha} \right) [f_{\alpha}(x^{\alpha})] \otimes \left[\left({}_{c}D_{x}^{\alpha} \right) [g_{\alpha}(x^{\alpha})] \right]^{\otimes -1}. \qquad \text{Q.e.d.}$$

Remark 3.2: In Theorem 3.1, the real number c can be replaced by c^{+}, c^{-}, $+\infty$, or $-\infty$.

Example 3.3: Suppose that $0 < \alpha \le 1$. Find the limit

$$\lim_{x \to 0^{+}} \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes Ln_{\alpha}(x^{\alpha}). \tag{27}$$

Solution $\lim_{x \to 0^{+}} \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes Ln_{\alpha}(x^{\alpha})$

$$= \lim_{x \to 0^{+}} Ln_{\alpha}(x^{\alpha}) \otimes \left[\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \right]^{\otimes -1}$$

$$= \lim_{x \to 0^{+}} \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \otimes \left[- \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -2} \right]^{\otimes -1} \text{ (by fractional L'Hospital's rule)}$$

$$= \lim_{x \to 0^{+}} - \frac{1}{\Gamma(\alpha+1)} x^{\alpha}$$

$$= 0 .$$

Example 3.4: Let $0 < \alpha \le 1$. Evaluate the limit

$$\lim_{x \to 0^{+}} \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes \frac{1}{\Gamma(\alpha+1)} x^{\alpha}}. \tag{28}$$

Solution

$$\lim_{x \to 0^{+}} \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes \frac{1}{\Gamma(\alpha+1)} x^{\alpha}}$$

$$= \lim_{x \to 0^{+}} E_{\alpha} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes Ln_{\alpha}(x^{\alpha}) \right)$$

$$= E_{\alpha} \left(\lim_{x \to 0^{+}} \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes Ln_{\alpha}(x^{\alpha}) \right)$$

$$= E_{\alpha}(0) \quad \text{(by Example 3.3)}$$

$$= 1 .$$

Example 3.5: If $0 < \alpha \le 1$, and $(-1)^{\alpha}$ exists. Find the limit

$$\lim_{x \to 0} \left[sin_{\alpha}(x^{\alpha}) - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right] \otimes \left[5 \cdot \frac{1}{\Gamma(3\alpha+1)} x^{3\alpha} \right]^{\otimes -1}. \tag{29}$$

Solution $\lim_{x \to 0} \left[sin_{\alpha}(x^{\alpha}) - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right] \otimes \left[5 \cdot \frac{1}{\Gamma(3\alpha+1)} x^{3\alpha} \right]^{\otimes -1}$

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$$= \lim_{x\to 0}[cos_\alpha(x^\alpha)-1]\otimes\left[5\cdot\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\right]^{\otimes -1}$$

$$= \lim_{x\to 0}[-sin_\alpha(x^\alpha)]\otimes\left[5\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes -1}$$

$$= \lim_{x\to 0}[-cos_\alpha(x^\alpha)]\otimes[5]^{\otimes -1}$$

$$= -\frac{1}{5}.$$

IV. CONCLUSION

The main purpose of this paper is to study the fractional L'Hospital's rule based on Jumarie's modified R-L fractional derivative. Using a new multiplication of fractional analytic functions and fractional Taylor series, we can easily prove the fractional L'Hospital's rule. On the other hand, we give three limit problems to illustrate its applications. In the future, we will continue to use fractional L'Hospital's rule to solve the problems in fractional calculus and engineering mathematics.

REFERENCES

[1] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics Vol.23, pp. 397-404, 2002.

[2] B. M. Vinagre and YangQuan Chen, Fractional calculus applications in automatic control and robotics, 41st IEEE Conference on decision and control Tutoral Workshop #2, Las Vegas, Desember 2002.

[3] J. Sabatier, OP Agrawal, JA Tenreiro machado, Advances in fractional calculus, Theoretical developments and applications in physics and engineering, Vol. 736 Springer; 2007.

[4] R. Hilfer, Ed., Applications of Fractional Calculus in Physics, World Scientific Publishing, Singapore,2000.

[5] F.Duarte and J. A. T. Machado, Chaotic phenomena and fractional-order dynamics in the trajectory control of redundant manipulators, Nonlinear Dynamics, Vol. 29, No. 1-4, pp. 315-342, 2002.

[6] M. da Grac¸a Marcos, F. B. M. Duarte, and J. A. T. Machado, Complex dynamics in the trajectory control of redundant manipulators, Nonlinear Science and Complexity, pp. 134-143, 2007.

[7] I. Petras, Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation. Springer, Berlin, 2011.

[8] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers, Vol. 1, Background and Theory, Vol 2, Application, Springer, 2013.

[9] H. A. Fallahgoul, S. M. Focardi and F. J. Fabozzi, Fractional calculus and fractional processes with applications to financial economics, theory and application, Elsevier Science and Technology, 2016.

[10] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering, Vol. 198, Academic Press, San Diego, USA, 1999.

[11] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, A Wiley-Interscience Publication, John Wiley & Sons, New York, USA, 1993.

[12] S. Samko, A. Kilbas, O. Marichev, Fractional Integrals and Derivatives, Gordon and Breach, 1993.

[13] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, INC. 1974.

[14] C. -H. Yu, Using trigonometric substitution method to solve some fractional integral problems, International Journal of Recent Research in Mathematics Computer Science and Information Technology, Vol. 9, No. 1, pp. 10-15, 2022.

[15] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, Vol. 3, No. 2, pp. 32-38, 2015.

[16] C. -H. Yu, Fractional Taylor series based on Jumarie type of modified Riemann-Liouville derivatives, International Journal of Latest Research in Engineering and Technology, Vol. 7, No. 6, pp.1-6, 2021.

[17] C. -H. Yu, A study on arc length of nondifferentiable curves, Research Inventy: International Journal of Engineering and Science, Vol. 12, No. 4, pp. 18-23, 2022.

[18] C. -H. Yu, Research on fractional exponential function and logarithmic function, International Journal of Novel Research in Interdisciplinary Studies, Vol. 9, No. 2, pp. 7-12, 2022.

[19] C. -H. Yu, A study on fractional derivative of fractional power exponential function, American Journal of Engineering Research, Vol. 11, No. 5, pp. 100-103, 2022.